Master of Science (Mathematics)
Semester – II

Paper Code –

PARTIAL DIFFERENTIAL EQUATIONS

Paper Code : 20MAT22C4
Partial Differential Equations

M. Marks = 100
Term End Examination = 80
Assignment = 20
Time = 3 Hours

Course Outcomes
Students would be able to:
CO1 Establish a fundamental familiarity with partial differential equations and their applications.
CO2 Distinguish between linear and nonlinear partial differential equations.
CO3 Solve boundary value problems related to Laplace, heat and wave equations by various methods.
CO4 Use Green's function method to solve partial differential equations.
CO5 Find complete integrals of Non-linear first order partial differential equations.

Section-I

Method of separation of variables to solve Boundary Value Problems (B.V.P.) associated with one dimensional Heat equation. Steady state temperature in a rectangular plate, Circular disc, Semi-infinite plate. The Heat equation in semi-infinite and infinite regions. Solution of three dimensional Laplace equations, Heat Equations, Wave Equations in Cartesian, cylindrical and spherical coordinates. Method of separation of variables to solve B.V.P. associated with motion of a vibrating string. Solution of Wave equation for semi-infinite and infinite strings. (Relevant topics from the book by O'Neil)

Section-II

Partial differential equations: Examples of PDE classification. Transport equation – Initial value problem. Non-homogeneous equations. Laplace equation – Fundamental solution, Mean value formula, Properties of harmonic functions, Green function.

Section-III

Heat Equation – Fundamental solution, Mean value formula, Properties of solutions, Energy methods. Wave Equation – Solution by spherical means, Non-homogeneous equations, Energy methods.

Section-IV

Non-linear first order PDE – Complete integrals, Envelopes, Characteristics, Hamilton Jacobi equations (Calculus of variations, Hamilton ODE, Legendre transform, Hopf-Lax formula, Weak solutions, Uniqueness).

Books Recommended:

- I.N. Sneddon, Elements of Partial Differential Equations, McGraw Hill, New York.
- Peter V. O'Neil, Advanced Engineering Mathematics, ITP.
- L.C. Evans, Partial Differential Equations: Second Edition (Graduate Studies in Mathematics) 2nd Edition, American Mathematical Society, 2010.
- H.F. Weinberger, A First Course in Partial Differential Equations, John Wiley & Sons, 1965. M.D. Raisinghania, Advanced Differential equations, S. Chand & Co.

Contents

CHAPTER-0

1 (a) Geometric Notations

(i) $R^n = n-$Dimensional real Euclidean space

(ii) $R^1 = R =$Real line

(iii) $e_i =$ Unit vector in the i^{th} direction $=(0,0,0,...1,...0)$

(iv) A point x in R^n is $x=(x_1,x_2,...,x_n)$

(v) $R^n = \{x=(x_1,x_2,...,x_n)\in R^n \,|x_n>0\}$ =open upper half-space

(vi) A point in R^{n+1} will be denoted as $(x,t)=(x_1,...,x_n,t)$, where t is time variable.

(vii) U,V,W denote open subsets of R^n .We write $V \subset\subset U$ if $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact i.e. V is compactly contained in U.

(viii) ∂U = boundary of U
U=closure of $U = U \cup \partial U$

(ix) $U_T = U \times (0,T]$

(x) $\Gamma_T = \bar{U}_T - U_T$ =parabolic boundary of U_T

(xi) $B^0(x,r)=\{y\in R^n \,\|x-y|<r\}$ = open ball in R^n with centre x and radius r>0

(xii) $B(x,r)=\{y\in R^n \,\|x-y|\le r\}$ =closed ball in R^n with centre x and radius r>0

(xiii) $\alpha(n)$ =volume of unit ball $B(0,1)$ in R^n

$$=\frac{r^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

$n\alpha(n)$ =surface area of unit sphere $B(0,1)$ in R^n

(xiv) If $a,b\in R^n$ s.t. $a=(a_1,a_2,...,a_n)$ and $b=(b_1,b_2,...,b_n)$ then $a,b=\sum_{i=1}^{n} a_i b_i$ and $|a|=\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$

(b) Notations for functions

(i) If $u:U\to R$,we write $u(x)=u(x_1,x_2,...,x_n)$ where $x\in U$, u is smooth if u is infinitely differentiable.

(ii) If u, v are two functions, we write $u\equiv v$ if u, v agree for all arguments
$u\coloneqq v$ means u is equal to v.

(iii) The support of a function u is defined as the set of points where the function is not zero and denoted by spt u.

$$u = \{x \in X \mid f(x) \neq 0\}$$

(iv) The sign function is defined by

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$u^{+} = \max(u, 0)$$
$$u^{-} = -\min(u, 0)$$
$$u = u^{+} - u^{-}$$
$$|u| = u^{+} + u^{-}$$

(v) If $u : U \to R^{m}$

$u(x) = \left(u^{1}(x), \ldots, u^{m}(x)\right)(x \in U)$ where $u = \left(u^{1}, u^{2}, \ldots u^{m}\right)$

The function u^{i} is the i$^{\text{th}}$ component of u

(vi) The symbol $\int_{\Sigma} f dS$ denotes the integral of f over $(n-1)$ dimensional surface $\sum$ in R^{n}

(vii) The symbol $\int_{C} f dl$ denotes the integral of f over the curve C in R^{n}

(viii) The symbol $\int_{V} f dx$ denotes the volume integral of S over $V \in R^{n}$ and $x \in V$ is an arbitrary point.

(ix) Averages: $\displaystyle \oint_{B(x,r)} f dy = \frac{1}{\alpha(n) r^{n}} \int_{B(x,r)} f dy$

=average of f over ball $B(x,r)$

$$\oint_{\partial B(n,r)} f ds = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(n,r)} f ds$$

=average of f over surface of ball $B(x,r)$

(x) A function $u : U \to R$ is called Lipschitz continuous if

$|u(x) - u(y)| \leq C|x - y|$, for some constant C and all $x, y \in U$ .We denote

$$Lip[u] = \sup_{\substack{x, y \in U \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|}$$

(xi) The convolution of functions f, g is denoted by $f * g$.

(c) Notations for derivatives: Suppose $u : U \to R, x \in U$

(i) $\dfrac{\partial u(x)}{\partial x_i} = \underset{h\to 0}{lt} \dfrac{u(x+he_i)-u(x)}{h}$

provided that the limit exists. We denote $\dfrac{\partial u}{\partial x}$ by u_{x_i}

Similarly $u_{x_i x_j} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$ and $u_{x_i x_j x_k} = \dfrac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}$ and in this way higher order derivatives can be defined.

(ii) **Multi-index Notation**

(a) A vector α of the for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ where each α_i is a non-negative integer is called a multi- index of order $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$

(b) For given multi-index α ,define

$$D^\alpha u(x) = \frac{\partial^\alpha u(x)}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

(c) If i is a non-negative integer

$$D^i u(x) = \{D^\alpha u(x), |\alpha| = i\}$$

The set of all partial derivatives of order i.

(d) $|D^k u(x)| = \left\{ \sum_{|\alpha|=k} |D^\alpha u|^2 \right\}^{1/2}$

(iii) $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$

=Laplacian of u

=trace of Hessian Matrix.

(iv) Let $x, y \in R^n i.e. x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n)$

Then we write

$$D_x u = (u_{x_1}, ..., u_{x_n})$$

$$D_y u = (u_{y_1}, ..., u_{y_n})$$

The subscript x or y denotes the variable w.r.t. differentiation is being taken

(d) Function Spaces

(i) (a) $C(U) = \{u : U \to R | \text{u is continous}\}$

(b) $C(\bar{U}) = \{u \in C(u) | u \ \ is \ \ uniformly \ \ continous \ on \ \ bounded \ \ subsets \ \ of \ U \}$

(c) $C^k(U) = \{u : U \to R | u \ \ is \ \ k \ \ times \ continuous \ \ differentiable\}$

(d) $C^k(\overline{U}) = \{u : C^k(U) \mid D^\alpha u$ is uniformly continuous unbounded subsets of U for all $|\alpha| \le k\}$

(e) $C^\infty(U) = \{u : U \to R \mid u \quad is \quad \inf initly \quad differentiable\}$

(ii) $C_c(U)$ means $C(U)$ has compact support.

Similarly, $C_c^k(U)$ means $C^k(U)$ has compact support.

(iii) The function $u : U \to R$ is Lebsegue measurable over L^p if $\|u\|_{L^p(U)} < \infty$

$$\|u\|_{L^p(U)} = \left(\int_U |u|^p \, dx\right)^{1/p}, 1 \le p \le \infty$$

The function $u : U \to R$ is Lebsegue measurable over L^∞ if $\|u\|_{L^\infty(U)} < \infty$

$$\|u\|_{L^\infty(U)} = ess \sup_U |u|$$

(iv) $L^p(U) = \{u : U \to R \mid u \quad is \quad Lebsegue \quad measurable\, over \quad L^p\}$

$L^\infty(U) = \{u : U \to R \mid u \quad is \quad Lebsegue \quad measurable\, over \quad L^\infty\}$

(v) $\|Du\|_{L^p(U)} = \||Du|\|_{L^p(U)}$

Similarly, $\|D^2u\|_{L^p(U)} = \||D^2u|\|_{L^p(U)}$

(vi) If $u : U \to R^m$ is a vector, where $u = (u^1, u^2, ..., u^m)$ then $D^k u = \{D^\alpha u, |\alpha| = k\}$

similarly other operator follow.

(e) Notation for estimates:

(i) Big Oh(O)order

We say

$f = O(g)$ as $x \to x_0$ provided there exists a constant C such that $|f(x)| \le C|g(x)|$, for all x sufficiently close to x_0.

(ii) Little Oh(o) order

We say

$f = o(g)$ as $x \to x_0$, provided $\lim_{x \to x_0} \left|\frac{f(x)}{g(x)}\right| \to 0$

2 Inequalities

(i) Convex Function

A function $f : R^n \to R$ is said to convex function if

$$f(\tau x + (1-\tau)y) \le \tau f(x) + (1-\tau) f(y)$$

for all $x, y \in R^n$ and each $0 < \tau < 1.$

(ii) Cauchy's Inequality

$$ab \le \frac{a^2}{2} + \frac{b^2}{2} \qquad (a, b \in R)$$

(iii) Holder's Inequality

Let $1 \le p, q \le \infty$; $\frac{1}{p} + \frac{1}{q} = 1$, $\qquad u \in L^p(u), v \in L^q(u)$

$$\int_U |uv|\, dx \le \|u\|_{L^p(U)} \|v\|_{L^q(U)}$$

(iv) Minkowski's Inequality

Let $1 \le p \le \infty$, and $u, v \in L^p(U)$, Then $\|u+v\|_{L^p(U)} \le \|u\|_{L^p(U)} + \|v\|_{L^q(U)}$

(v) Cauchy Schwartz Inequality

$$|x.y| < |x||y| \qquad (x, y \in R^n)$$

3 Calculus

(a) Boundaries

Let $U \subset R^n$ be open and bounded, k={1,2,…,}

Definitions:

(i) The boundary ∂U is C^k if for each point $x^0 \in \partial U$ there exists r>0 and a C^k function $\Upsilon : R^{n-1} \to R$ such that $U \cap B(x^0, r) = \{x \in B(x^0, r) | x_n > \Upsilon(x_1, ..., x_{n-1})\}$

Also, ∂U is analytic if Υ is analytic.

(ii) If U is C^1, then along ∂U, the outward unit normal at any point $x_0 \in \partial U$ is denoted by $\underline{v}(x^0) = (v_1, ..., v_n)$.

(iii) Let $u \in C^1(\bar{U})$ then normal derivative of u is denoted by $\frac{\partial u}{\partial v} = \underline{v}.Du$

(b) Gauss-Green Theorem

Let U be a bounded open subset of R^n and ∂U is C^1. $u : U \to R^n$ and also $u \in C^1(\bar{U})$ then

$$\int_U u_{x_i}\, dx = \int_{\partial U} uv^i\, dS \qquad (i = 1, 2, ..., n)$$

(c) Integration by parts formula

Let $u, v \in C^1(\bar{U})$ then

$$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} u v \nu^i dS$$

Proof: By Gauss-Green's Theorem

$$\int_U (uv)_{x_i} dx = \int_{\partial U} (uv) \nu^i dS$$

Or $$\int_U u_{x_i} v dx + \int_U u v_{x_i} dx = \int_{\partial U} (uv) \nu^i dS$$

Or $$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} (uv) \nu^i dS$$

(d) Green's formula

Let $u, v \in C^2(\bar{U})$ then

(i) $\int_U \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$

Proof: $\int_U \Delta u dx = \int (u_{x_i})_{x_i} dx$

Integrating by parts, taking the second function as unity

$$\int_U \Delta u dx = \int_{\partial U} u_{x_i} \nu^i dS$$

$$= \int_{\partial U} \frac{\partial u}{\partial \nu} dS$$

Hence proved.

(ii) $\int_U Du.Dv dx = -\int_U u \Delta v dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u dS$

Proof: $\int_U Du.Dv dx = -\int_U u \Delta v dx + \int_{\partial U} u Dv.\nu dS$

$$= -\int_U u \Delta v dx + \int_{\partial u} u \frac{\partial v}{\partial \nu} dS \qquad \text{(integrating by parts)}$$

(iii) $\int_U (u \Delta v - v \Delta u) dx = \int_{\partial U} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) dS$

Proof: $\int_U u \Delta v dx = -\int_U Du.Dv dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u dS$

Similarly, $\int_U v \Delta u dx = -\int_U Du.Dv dx + \int_{\partial U} \frac{\partial u}{\partial \nu} u dS$

subtracting, we get the result.

(e) Conversion of n-dimensional integrals into integral over sphere

(i) Coarea formula

Let $u : R^n \to R$ be Lipschitz continuous and assume that for a.e. $r \in R$,the level set $\{x \in R^n \,|u(x) = r\}$ is a smooth and n-1 dimensional surface in R^n .Suppose also $f : R^n \to R$ is smooth and summable. Then

$$\int_{R^n} f\,|Du|\,dx = \int_{-\infty}^{\infty}\left(\int_{\{u=r\}} fdS\right)dr$$

Cor. Taking $u(x) = |x - x_0|$

Let $f : R^n \to R$ be continuous and summable then

$$\int_{R^n} fdx = \int_{0}^{\infty}\left(\int_{\partial B(x_0,r)} fdS\right)dr$$

for each point $x_0 \in R^n$ or we can say

$$\frac{d}{dr}\left(\int_{B(x_0,r)} fdx\right) = \int_{\partial B(x_0,r)} fdS$$

for each r>0.

(f) To construct smooth approximations to given functions

Def: If $U \subset R^n$ is open, given $\varepsilon > 0$ .We define $U_\varepsilon := \{x \in U \,|dist(x, \partial U) > \varepsilon\}$

Def. Standard Mollifier

Let $\eta \in C^\infty(R^n)$ such that

$$\eta(x) := \begin{cases} c\exp\left(\dfrac{1}{|x|^2 - 1}\right) if \quad |x| < 1 \\ 0\, if \quad |x| > 1 \end{cases}$$

The constant c is chosen so that $\int_{R^n} \eta dx = 1$

Def. We define

$\eta_\varepsilon(x) := \dfrac{1}{\varepsilon^n}\eta\left(\dfrac{x}{\varepsilon}\right)$ for every $\varepsilon > 0$.

Properties:

(i) The functions η_ε are C^∞ since $\eta(x)$ are C^∞ .

(ii) $\int_{R^n} \eta_\varepsilon dx = \dfrac{1}{\varepsilon^n}\int_{R^n} \eta\left(\dfrac{x}{\varepsilon}\right)dx$

$= \int_{R^n} \eta(x)\,dx$ (by definition of n-tuple integral)

$=1$

(g) Mollification of a function

If $f:U \to R$ is locally integrable

We define the mollification of f

$f^{\varepsilon} := \eta_{\varepsilon} * f$ in U_{ε}

$$= \int_U \eta_{\varepsilon}(x-y) f(y) dy \quad = \int_{B(0,\varepsilon)} \eta_{\varepsilon}(y) f(x-y) dy \qquad \text{(by definition)}$$

Properties:

(i) $f^{\varepsilon} \in C^{\infty}(U_{\varepsilon})$

(ii) $f^{\varepsilon} \to f$ almost everywhere as $\varepsilon \to 0$

(iii) If $f \in C(U)$ then $f^{\varepsilon} \to f$ uniformly on compact subset of U almost everywhere.

Function Analysis Concepts

(i) L^p **space**: Assume U to be a open subset of R^n and $1 \leq p \leq \infty$ .If $f:U \to R$ is measurable, we define

$$\|f\|_{L^p(U)} := \begin{cases} \left(\int_U |f|^p dx\right)^{1/p} & if \quad 1 \leq p < \infty \\ ess\sup_U |f| & if \quad p = \infty \end{cases}$$

Transformation from Ball $B(x,r)$ **to unit Ball** $B(0,1)$

Let $B(x,r)$ be a ball with centre x and radius r and $B(0,1)$ be an arbitrary point of $B(x,r)$ and z be an arbitrary point of $B(0,1)$ then relation between y and z is y=x+rz.